Discrete Random Variables

A r.v. $X: \Omega \rightarrow \mathbb{R}$ is discrete if there exists a set $S_X = \{x_1, x_2, x_3, \ldots\}$ such that for all $\omega \in \Omega$, $X(\omega) \in S_X$. If $S_X$ is a finite set, then $X$ is a finite r.v.

Since $S_X$ is discrete, we can consider events in $\Omega$ of the form $A_k = \{\omega \in \Omega \mid X(\omega) = x_k\}$. These events will form a partition of $\Omega$.

In fact, all events involving $X$ can be computed from the $P[A_k]$.

The probability mass function (PMF) of a discrete r.v. $X$ is

$$p_X(x) = \begin{cases} P[X = x] = P[\{\omega \in \Omega : X(\omega) = x\}] = \begin{cases} P[A_k] & \text{if } x = x_k \\ 0 & \text{otherwise} \end{cases} & \text{if } x = x_k \end{cases}$$
The function \( p : \mathbb{R} \to \mathbb{R} \) is a valid PMF iff

1. the support of \( p \) is discrete, \( S_x = \{ x \in \mathbb{R} : p(x) > 0 \} \)
2. \( p(x) \geq 0 \), \( \forall x \in \mathbb{R} \),
3. \( \sum_{x \in S_x} p(x) = 1 \).

The CDF of \( X \) if \( X \) is discrete w/ PMF \( p_x(.) \) is

\[
F_X(x) = P[X \leq x] = \sum_{x_k \in S_x} p(x_k) \quad x_k \leq x
\]

If \( u(x) \) is the right continuous step function: \( u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \),

then \( F_X(x) = \sum_{x_k \in S_x} p(x_k) u(x-x_k) \).
To completely describe discrete $X$, requires the PMF or the CDF.

Sometimes, need to **summarize** distribution of $X$.

The mean or expected value of $X$ is

$$M_X = E[X] = \sum_{x \in X} x P_X(x) = \sum_k x_k P_X(x_k).$$

This will be well defined provided that $\sum_k |x_k| P_X(x_k) < \infty$.

If $X$ is discrete and $g: \mathbb{R} \rightarrow \mathbb{R}$, then $Z = g(X)$ is also a discrete r.v.

We can write the mean of $Z$ w/o explicit reference to the distribution of $Z$.

$$E[Z] = \sum_{x \in X} g(x_k) P_X(x_k).$$

The distribution of $X - E[X]$ would tell us how much $X$ deviates from its mean. $E[X - E[X]] = 0$. Instead, could look at $|X - E[X]|$. Instead, usually look at $(X - E[X])^2$. 

2015-09-03 p3
\[ \sigma_x^2 = \text{Var}[X] = E[(X - E[X])^2] \quad \text{(Variance)} \]

\[ \sigma_x = \sqrt{\text{Var}[X]} \quad \text{(Standard Deviation) \ (Same units as } X) \]

Expectation is \underline{linear}.

\[ E[aX + bY] = aE[X] + bE[Y] \]

\[ \text{VAR}[X] = E[(X - E[X])^2] = E[X^2 - 2E[X]X + (E[X])^2] \]

\[ = E[X^2] - 2E[X]E[X] + \underbrace{E[(E[X])^2]}_{2\text{nd Moment of } X} \]

\[ = E[X^2] - (E[X])^2 \quad \text{\uparrow \ mean\ of \ } X \]
Bernoulli RV

Let A be an event defined on $\Omega$.

The Bernoulli RV is

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise.} \end{cases}$$

If $X$ is Bernoulli....

$$p_X(x) = \begin{cases} p[A] & x = 1 \\ p[A^c] = 1-p[A] & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

1. Let $\rho = p[A]$.

   The Bernoulli r.v. has

   one parameter, $\rho$.

2. CDF

   $$F_X(x) = (1-\rho)u(x) + \rho u(x-1)$$

3. $E[X] = \sum_{x \in S_X} x p_X(x) = 0 \cdot p[A^c] + 1 \cdot p[A] = p[A] = p[X=1] = \rho$.

2015-09-03 $\rho$5
Binomial RV

Number of successes in \( n \) independent trials.

Two parameters: \( p = \text{P}[A] \) for one trial, \( n = \# \text{ of trials} \)

\( S_X = \{0, 1, 2, \ldots, n\} \)

\[ P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{for } k \leq 0, 1, 2, \ldots, n \geq 3 \\ 0 & \text{otherwise.} \end{cases} \]

\( \binom{n}{k} \) is called the \underline{binomial coefficient} because

\[ (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} . \]

\[ \sum_{k=0}^{n} P_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1^n = 1 . \]
\[ E[X] = \sum_{k=0}^{n} \binom{n}{k} \rho^k (1-\rho)^{n-k} = \sum_{k=1}^{n} \frac{n!}{k! (n-k)!} \rho^k (1-\rho)^{n-k} \]

\[ = n\rho \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)! (n-k)!} \rho^{k-1} (1-\rho)^{n-k} \]

\[ = n\rho \sum_{k'=0}^{n-1} \frac{(n-1)!}{k'! (n-1-k')!} \rho^{k'} (1-\rho)^{n-1-k'} \]

\[ = n\rho \sum_{k'=0}^{n-1} \binom{n-1}{k'} \rho^{k'} (1-\rho)^{n-1-k'} \]

PMF of a binomial

\[ n-1, \rho \]

Must sum to 1

**Variance**

\[ Var(X) = \sum_{k=0}^{n} k^2 \binom{n}{k} \rho^k (1-\rho)^{n-k} = n\rho \sum_{k=0}^{n} k^2 \frac{n!}{(n-k)! k!} \rho^k (1-\rho)^{n-k} \]

\[ = n\rho \sum_{k'=0}^{n-1} (k'+1) \binom{n-1}{k'} \rho^{k'} (1-\rho)^{n-1-k'} \]

\[ = n\rho \left[ \sum_{k'=0}^{n-1} k' \binom{n-1}{k'} \rho^{k'} (1-\rho)^{n-1-k'} + \binom{n-1}{k'} \rho^{n-1-k'} \right] \]

\[ = n\rho \left[ (n-1)\rho + 1 \right] = n\rho (n\rho + (1-\rho)) \]
\[ \text{Var}[X] = E[X^2] - (E[X])^2 \]
\[ = np(np + (1-p)) - (np)^2 \]
\[ = np(1-p) \]

**Poisson**

Events that occur at an average rate \( \lambda \) events per time interval, events occur "at random".

If \( X \) is the number of events that occur in one time interval then \( X \) is a Poisson r.v.

\[ S_X = \{0,1,2,3,\ldots\} \]

\[ p_X(k) = \begin{cases} \frac{\lambda^k e^{-\lambda}}{k!} & k \in \{0,1,2,\ldots\} \\ 0 & \text{otherwise.} \end{cases} \]

Poisson also used as an approximation of the binomial, when \( p \) is small and \( n \) is large (take \( \lambda = np \)).

2015-09-03 P3
\[ \sum_{k=0}^{\infty} p_X(k) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \]

Taylor Series expansion for \( e^x \)

\[ = e^{-\lambda} \cdot e^\lambda = 1 \]

\[ E[X] = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^{k'}}{k'!} = \lambda \]

PMF of Poisson

\[ E[X^2] = \sum_{k=0}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} = \lambda \sum_{k=1}^{\infty} k e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!} \]

\[ = \lambda (\lambda + 1) = \lambda^2 + \lambda \]

\[ \text{Var}[X] = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda \]
Ex Call center. Calls arrive “at random” with an average rate of 4 calls/minute.

(a) What is \( P[\text{more than 4 calls in 10 sec}] \)?
(b) What is \( P[\text{fewer than 5 calls in 2 min}] \)?

(a) \( P[\text{more than 4 calls /10 sec}] \).

For 10 second intervals,
\[
\lambda = \left(4 \text{ calls/min} \right) \left( \frac{1 \text{ min}}{60 \text{ sec}} \right) (10 \text{ sec}) = \frac{2}{3} \text{ call/10 sec}
\]

\[
P[X > 4] = 1 - P[X \leq 4] = 1 - \sum_{k=0}^{4} \frac{\left(\frac{2}{3}\right)^k}{k!} e^{-2/3} = 6.3 \times 10^{-4}
\]

(b) 2 min \( \rightarrow \lambda = 8 \text{ calls/2 min} \)

\[
P[X \leq 5] = \sum_{k=0}^{5} \frac{8^k}{k!} (e^{-8}) = 0.10
\]