$\Omega$ - Sample Space

$F$ is a set of subsets of $\Omega$.

$F$ is a $\sigma$-field:

1. $F$ is non-empty.
2. $F$ is closed under complements: If $A \in F$, then $A^c \in F$.
3. $F$ is closed under countable unions:
   If $A_1, A_2, \ldots \in F$ then $\bigcup_{k=1}^{\infty} A_k \in F$.

A probability law is a function,

$\rho: F \rightarrow \mathbb{R}$.

Consequences of the fact that $F$ is a $\sigma$-field:

1. $F$ must contain $\Omega$ and $\emptyset$.
2. $F$ is closed under countable (and finite) intersections.
3. The set of all subsets of $\Omega$ is a $\sigma$-field.
\( P \) is a probability law iff

1. \( P[A] \geq 0 \quad \forall \ A \in \mathcal{F} \)
2. \( P[\Omega] = 1 \)
3. If \( A_1, A_2, \ldots \) is a sequence of disjoint events (that is \( A_i \cap A_j = \emptyset \) if \( i \neq j \)), then
   \[
   P \left[ \bigcup_{k=1}^{\infty} A_k \right] = \sum_{k=1}^{\infty} P[A_k].
   \]

If \( \Omega \) is finite, the (3) can be simplified to

3'. If \( A_1 \cap A_2 = \emptyset \) then
   \[
   P[A_1 \cup A_2] = P[A_1] + P[A_2].
   \]

Consequences:

- \( P[A^c] = 1 - P[A] \)
\[- P[A \cup B] = P[A] + P[B] - P[A \cap B] \]

\[A \cup B = (A \cap B^c) \cup (A \cap B) \cup (B \cap A^c)\]

\[A = (A \cap B^c) \cup (A \cap B)\]

\[B = (B \cap A^c) \cup (B \cap A)\]

\[\text{By axiom 3:}\]

\[P[A \cup B] = P[A \cap B^c] + P[A \cap B] + P[B \cap A^c]\]

\[P[A] = \underbrace{P[A \cap B^c]} + P[A \cap B]\]

\[P[B] = \underbrace{P[B \cap A^c]} + P[B \cap A]\]

\[P[A \cap B] = P[A] - P[A \cap B^c]\]

\[P[B] = P[B] - P[B \cap A^c]\]

\[P[A \cup B] = \frac{P[A \cap B]}{P[A] + P[B] - P[A \cap B]}\]

\[= P[A] + P[B] - P[A \cap B]\]
- If $A \subseteq B$ then $P[A] \leq P[B]$.

$$B = A \cup (B \cap A^c)$$

$$P[B] = P[A] + P[B \cap A^c] \geq P[A]$$

A probability space is an ordered tuple $(\Omega, \mathcal{F}, P)$.

If $\Omega$ is finite or countable, then probability space is discrete.

In this case can often take $\mathcal{F}$ as set of all subsets of $\Omega$.

Then $P$ completely determined by $P[\{\omega\}]$ for all $\omega \in \Omega$.

If $\Omega$ is uncountable, probability space is continuous.

In this case $\mathcal{F}$ cannot be the set of all subsets of $\Omega$.

Most often, let $\mathcal{F}$ be the "Borel $\sigma$-field" over $\Omega$.

The Borel $\sigma$-field consists of all open and closed intervals of the real line and anything formed by countable unions of those intervals.
A random experiment has a procedure and observations. The possible observations define the sample space, $S$. In some cases, can choose observations such that every outcome $w \in S$ is equiprobable.

Roll die twice
- Observe sum $S_2 = \{2, 3, 4, \ldots, 12\}$
- Observe individual rolls: $S = \{(1,1), (1,2), (1,3), \ldots, (6,6)\}$

If all elements $w \in S_2$ are equiprobable, then for any $A \subset S_2$,

$$P[A] = \frac{|A|}{|S_2|}$$

(where $|A|$ is the cardinality of the set)

**Multiplication Rule**

Wish to construct a k-tuple with 1 item each from sets containing $n_1, n_2, \ldots, n_k$ items.

There are $n_1 \cdot n_2 \cdots n_k$ choices.
Ex. Deli: 2 sizes, 4 breads, 5 proteins, 3 cheese.
    # of sandwiches = 2·4·5·3 = 120.

**Permutation Rule**

Number of ways to choose \( k \) items, in order, from a set of \( n \) items without replacement.

\[
P^n_k = n \cdot (n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}
\]

**Quick Note**

\[
n! = \prod_{k=1}^{n} k
\]

Therefore, \( 0! = 1 \).

**Standard Deck of Playing Cards**

52 cards, each labeled w/ a suit: spades, clubs, hearts, diamonds and a value: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

How many ways to deal 5 cards (order matters)?

\[
P^{52}_{5} = \frac{52!}{47!} = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 3,113,785,200
\]
Combination Rule

Number of ways to choose \( k \) items from a set of \( n \) items without replacement (where order of selection doesn't matter).

\[
\binom{n}{k} = \frac{p^n_k}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}
\]

A poker hand is 5 cards from a standard deck.

How many poker hands?

\[
\binom{52}{5} = \frac{52!}{47! \cdot 5!} = 2,598,960
\]

Examples

1. How many distinct arrangements of 6 (identical) red balls and 6 (identical) blue balls?

\[
\frac{12!}{6! \cdot 6!} = \binom{12}{6}
\]
2. Suppose a shipment of 50 widgets contains 10 defective widgets.
   a. If 5 are tested, what is the probability that none of the defective widgets are tested?
   b. What if 10 are tested?
   c. If 10 are tested, what is the probability that no more than 1 is defective?

(a) \( \frac{\binom{40}{5}}{\binom{50}{5}} = 0.31 \)

(b) \( \frac{\binom{40}{10}}{\binom{50}{10}} = 0.08 \)

(c) \( \frac{\binom{10}{1}\binom{40}{9} + \binom{40}{10}}{\binom{50}{10}} = 0.35 \)
Conditional Probability

The conditional probability of event A given event B, written $P[A|B]$, is the probability that event A occurs when event B is known to have occurred.

If $P[B] > 0$, then

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

Note that $P[A|B]$ is a probability rule.

Can rearrange to


Fair, six-sided die tossed twice, record each roll.

A: the total of the two rolls is even
B: each of the two tosses is even.

$B \subset A \implies P[A \cap B] = P[B]$
Theorem on Total Probability + Bayes' Rule

Let \( B_1, B_2, B_3, \ldots, B_n \) be a set of mutually exclusive events \((B_i \cap B_j = \emptyset \text{ if } i \neq j)\) such that \( \bigcup_{k=1}^{n} B_k = \Omega \). Such a set is called a partition of the sample space.

Now, let \( A \) be any event, then

\[
A = A \cap \Omega = A \cap (B_1 \cup B_2 \cup \cdots \cup B_n)
\]

\[
= (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)
\]

\[
P[A] = P[A \cap B_1] + P[A \cap B_2] + \cdots + P[A \cap B_n]
\]

\[
= P[A \mid B_1]P[B_1] + P[A \mid B_2]P[B_2] + \cdots + P[A \mid B_n]P[B_n]
\]

Theorem on Total Prob.